INTRODUCTION TO QUANTUM INFORMATION SCIENCE

Problem Sheet 4
Hilary Term

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Questions Label: A - Bookwork B - Standard C - Challenging/Optional

4.1.B **Completely positive maps.** Any physically admissible operation on a qubit is described by a completely positive map which can always be written as

$$\varrho \mapsto \varrho' = \sum_k A_k \varrho A_k^\dagger,$$

where matrices A_k satisfy $\sum_k A_k^\dagger A_k = \mathbb{1}$.

(1) Show that this map preserves positivity and trace. Show that any composition of completely positive maps is also completely positive.

(2) A qubit in state ϱ is transmitted through a depolarising channel that effects a completely positive map

$$\varrho \mapsto (1-p)\varrho + \frac{p}{3}\left(\sigma_x \varrho \sigma_x + \sigma_y \varrho \sigma_y + \sigma_z \varrho \sigma_z\right),$$

for some $0 \leq p \leq 1$. Show that under this map the Bloch vector associated with ϱ shrinks by the factor $(3-4p)/3$.

4.2.B **Positive but not completely positive maps.** Consider a map $\mathcal{N}$, called universal-NOT, which acts on single qubit density matrices and is defined by its action on the identity and the three Pauli matrices

Any 2×2 matrix can be written as a linear composition of the identity and the three Pauli matrices as discussed in Question 1.1.

$$\mathcal{N}(\mathbb{1}) = \mathbb{1} \quad \mathcal{N}(\sigma_x) = -\sigma_x \quad \mathcal{N}(\sigma_y) = -\sigma_y \quad \mathcal{N}(\sigma_z) = -\sigma_z$$

(1) Describe the action of this map in terms of the Bloch vectors.

(2) Explain why $\mathcal{N}$, acting on a single qubit, maps density matrices to density matrices.

(3) The joint state of two qubits is described by the density matrix

$$\rho = \tfrac{1}{4}\left(\mathbb{1}\otimes\mathbb{1} + \sigma_x\otimes\sigma_x - \sigma_y\otimes\sigma_y + \sigma_z\otimes\sigma_z\right),$$

Apply $\mathcal{N}$ to the first qubit leaving the second qubit intact. Write the resulting matrix and explain why $\mathcal{N}$ is not a completely-positive map.

4.3.B **Approximate cloning.** Consider a hypothetical universal quantum cloner that operates on two qubits and on some auxiliary system. Given one qubit in any quantum state $|\psi\rangle$ and the other one in a prescribed state $|0\rangle$ it maps

$$|\psi\rangle\,|0\rangle\,|R\rangle \mapsto |\psi\rangle\,|\psi\rangle\,|R'\rangle,$$

where $|R\rangle$ and $|R'\rangle$ are, respectively, the initial and the final state of any other auxiliary system that may participate in the cloning process ($|R'\rangle$ may depend on $|\psi\rangle$).

(1) Show that such a cloner is impossible.

But supposed we are willing to settle for an imperfect copy? It turns out that the best approximation to the universal quantum cloner is the following transformation

$$|\psi\rangle\,|0\rangle\,|0\rangle \mapsto \sqrt{\frac{2}{3}}\,|\psi\rangle\,|\psi\rangle\,|\psi\rangle + \sqrt{\frac{1}{6}}\left(|\psi\rangle\left|\psi^\perp\right\rangle + \left|\psi^\perp\right\rangle|\psi\rangle\right)\left|\psi^\perp\right\rangle$$

where $\left|\psi^\perp\right\rangle$ is a normalised state vector orthogonal to $|\psi\rangle$ and the auxiliary system is another qubit.

(2) Given the transformation above explain why the reduced density matrices of the first and the second qubit must be identical after the transformation.

(3) Show that the reduced density matrix of the first (and the second) qubit can be written as

$$\rho = \frac{5}{6}\,|\psi\rangle\langle\psi| + \frac{1}{6}\left|\psi^\perp\right\rangle\left\langle\psi^\perp\right|.$$

(4) What is the probability that the clone in state ρ will pass a test for being in the original state $|\psi\rangle$?

(5) What is the relation between the Bloch vectors of $|\psi\rangle\langle\psi|$ and ρ?

4.4.B **CP maps revisited.** Any linear transformation (superoperator) T acting on density matrices of a qubit can be completely characterised by its action on the four basis matrices $|a\rangle\langle b|$, where $a, b = 0, 1$, and can be represented as a 4×4 matrix,

$$\tilde{T} = \left[\begin{array}{c|c} T(|0\rangle\langle 0|) & T(|0\rangle\langle 1|) \\ \hline T(|1\rangle\langle 0|) & T(|1\rangle\langle 1|) \end{array}\right].$$

Write down $\tilde{T}$ for:

(1) transposition, $\varrho \mapsto \varrho^T$,

(2) depolarising channel, $\varrho \mapsto (1-p)\varrho + \frac{p}{3}\left(\sigma_x\varrho\sigma_x + \sigma_y\varrho\sigma_y + \sigma_z\varrho\sigma_z\right)$, for some $0 \le p \le 1$.

Show that for completely positive maps T matrix $\tilde{T}$ must be positive semidefinite.

4.5.B **Quantum error correction.**

(1) Draw a quantum network (circuit) that encodes a single qubit state $\alpha\,|0\rangle + \beta\,|1\rangle$ into the state $\alpha\,|00\rangle + \beta\,|11\rangle$ of two qubits. Here and in the following α and β are some unknown generic complex coefficients.

(2) Two qubits were prepared in state $\alpha\,|00\rangle + \beta\,|11\rangle$, exposed to bit flip-errors, and then measured with an ancillary qubit, as shown in Fig. 1. The result of the measurement is x. Can you infer the absence of errors when $x = 0$? Can you infer the presence of errors when $x = 1$? Can you correct any detected errors?

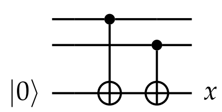


Fig.1

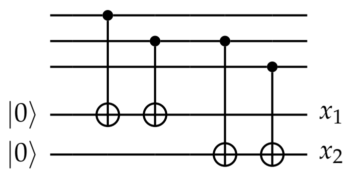


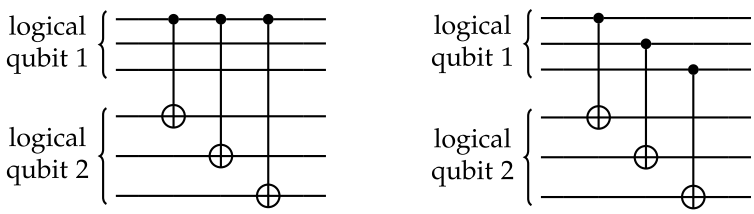
Fig.2

Three qubits were prepared in state $\alpha\,|000\rangle + \beta\,|111\rangle$ and then, by mistake, someone applied the Hadamard gate to one of them, but nobody remembers which one. Your task is to recover the original state of the three qubits.

(3) Express the Hadamard gate as the sum of two Pauli matrices. Pick up one of the three qubits and apply the Hadamard gate. How is the state $\alpha |000\rangle + \beta |111\rangle$ modified? Interpret this in terms of bit-flip and phase-flip errors.

(4) You perform the error syndrome measurement shown in Fig. 2. Suppose the outcome of the measurement is $x_1 = 0, x_2 = 1$. How would you recover the original state? Describe the recovery procedure when $x_1 = 0, x_2 = 0$.

The figure below shows two implementations of a controlled-NOT gate acting on the encoded states of the three qubit code.



Implementation A Implementation B

(5) Assume that the only sources of errors are individual controlled-NOT gates which produce bit-flip errors in their outputs. These errors are independent and occur with a small probability p. For each of the two implementations find the probability of generating unrecoverable errors at the output. Which of the two implementations is fault-tolerant?

4.6.B **Stabilisers define vectors and subspaces.**

In Problem sheet 1, we have discuss the concept of 1-qubit Pauli group and also the concept of stabiliser groups. Here we will further explore these concepts.

The n-qubit Pauli group is defined as

$$\mathbb{G}_n = \{\mathbb{1}, X, Y, Z\}^{\otimes n} \otimes \{\pm 1, \pm i\}$$

where X, Y, Z are the Pauli matrices. Each element of $\mathbb{G}_n$ is, up to an overall phase ± 1, $\pm i$, a tensor product of Pauli matrices and identity matrices acting on the n qubits.

A unitary S stabilises $|\psi\rangle$ if $S|\psi\rangle = |\psi\rangle$ and we have shown in Problem sheet 1 that the set of stabilisers of a given state $|\psi\rangle$ forms a group (known as the stabiliser group). As we will see later, we will generalise the concept of stabiliser groups from stabilising a state to stabilising a subspace (i.e. stabilising all states in the subspace), which is called a *code space*. We shall restrict our attention to stabiliser groups $\mathbb{S}$ that are subgroups of $\mathbb{G}_n$.

(1) Explain why in order to have a non-trivial (non-zero-dimension) code space, the stabiliser group must be Abelian (i.e. all of its elements commute) and do not contain the element $-\mathbb{1}$?

(2) Explain why all such stabilisers (except the identity $\mathbb{1}$) have trace zero and square to $\mathbb{1}$.

(3) Show that each stabiliser S has the same number of eigenvectors with eigenvalues $+1$ and -1, and hence "splits" the 2^{2n} dimensional Hilbert space in half. How would you describe the action of the two operators $\frac{1}{2}(\mathbb{1} \pm S)$?

(4) Consider two stabiliser generators, S_1 and S_2. Show that eigenvalue $+1$ subspace of S_1 is split again in half by S_2. That is, in that subspace exactly half of the S_2 eigenvectors have eigenvalue $+1$ and the other half -1.

Hint: Show that $\mathrm{tr}\, \frac{1}{2}(\mathbb{1} + S_1) S_2 = 0$.

(5) If a stabiliser group in the Hilbert space of dimension 2^n has a minimal number of generators, $S_1, \ldots, S_r$, what is dimension of the stabiliser subspace?

(6) State $|0\rangle$ is stabilised by Z and state $|1\rangle$ is stabilised by $-Z$. What are stabiliser generators for the standard basis of two qubits, i.e. for the states $|00\rangle$, $|01\rangle$, $|10\rangle$ and $|11\rangle$? What are stabiliser generators for each of the four Bell states?

(7) Construct stabiliser generators for an $n = 3$, $k = 1$ (n physical qubits encoding k logical qubits) code that can correct a single bit flip, i.e. ensure that recovery is possible for any of the errors in the set $\mathcal{E} = \{\mathbb{1}\mathbb{1}\mathbb{1}, X\mathbb{1}\mathbb{1}, \mathbb{1}X\mathbb{1}, \mathbb{1}\mathbb{1}X\}$. Find an orthonormal basis for the two-dimensional code subspace.

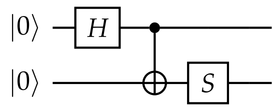
We often drop the tensor product symbol, e.g. $\mathbb{1} \otimes X \otimes \mathbb{1} \equiv \mathbb{1}X\mathbb{1}$. For commonly used single-qubit gates, sometimes we simply use subscripts to denote which qubits they acts on, e.g. $\mathbb{1} \otimes X \otimes \mathbb{1} \equiv X_1$ or $X \otimes \mathbb{1} \otimes Z \equiv X_1 Z_3$.

(8) Describe the subspace fixed by the stabiliser generators $X \otimes X \otimes \mathbb{1}$ and $\mathbb{1} \otimes X \otimes X$ and its relevance for quantum error correction.

(9) Let S_1 and S_2 be stabiliser generators for a two qubit state $|\psi\rangle$. The state is modified by a unitary operation U. What are the stabiliser generators for $U|\psi\rangle$?

(10) Step through the circuit

Here S is a phase gate: $|0\rangle \mapsto |0\rangle$ and $|1\rangle \mapsto i\,|1\rangle$.



writing down quantum states of the two qubits after each unitary operation and their respective stabiliser generators. How would you describe the action of the three gates, H, S and controlled-NOT, in the stabiliser language?

4.7.B **Shor's 9-qubit code.** Use 8 stabiliser generators for Shor's 9-qubit code and explain why this code can correct an arbitrary single-qubit error. In fact, it can also correct some multiple-qubit errors. Which of the following errors can be corrected by the nine-qubit code: X_1X_3, X_2X_7 , X_5Z_6 , Z_5Z_6 , Y_2Z_8?

X_i, Y_i, or Z_i represents X, Y, or Z applied to the i-th qubit.