INTRODUCTION TO QUANTUM INFORMATION SCIENCE

Problem Sheet 2
Hilary Term

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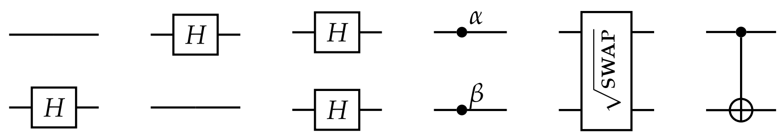
Questions Label: A - Bookwork B - Standard C - Challenging/Optional

2.1.B **Two-qubit operations.** The circuits below show six unitary operations on two qubits,

$P(\alpha) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$

The square root of SWAP matrix has something in common with the square root of NOT. Start with writing the SWAP matrix.

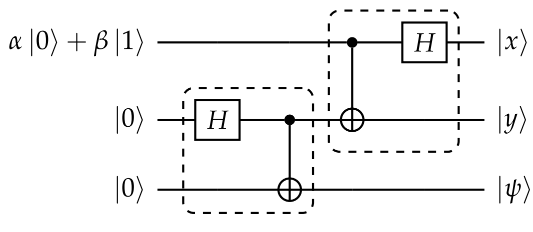


The first four are described, respectively, by 4×4 unitary matrices which are tensor products $\mathbb{1} \otimes H$, $H \otimes \mathbb{1}$, $H \otimes H$ and $P(\alpha) \otimes P(\beta)$. The matrices of the two remaining gates, known as the square root of SWAP and controlled-NOT, stand out as they do not admit a tensor product decomposition in terms of single-qubit operations. Use the standard tensor product basis, $|00\rangle$, $|01\rangle$, $|10\rangle$, $|11\rangle$, and write down unitary matrices for each of the six gates.

2.2.B **Basic entanglement.** Prove that the state of two qubits $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$ is entangled iff $ad - bc \neq 0$. Deduce that the state $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + (-1)^k |11\rangle)$ is entangled for $k = 1$ and unentangled for $k = 0$. Express the latter case explicitly as a product state.

2.3.A **Quantum teleportation.** Consider the following quantum network (circuit), containing the Hadamard and the controlled-NOT gates,

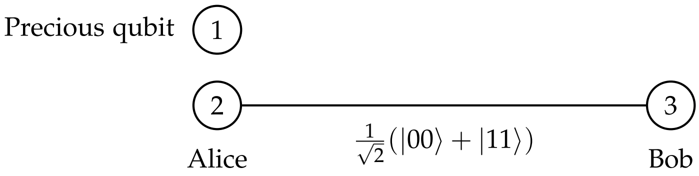
You should remember the action of the Hadamard and the controlled-NOT gates.



Divide et impera, that is, divide and conquer, a good approach to solving problems in mathematics (and in life). Start with smaller circuits, those surrounded by the dashed boxes.

The measurement on the first two qubits (counting from the top) gives two binary digits, x and y. The third qubit is not measured. How does the state of the third qubit, $|\psi\rangle$, depend on the values x and y?

Suppose the three qubits, which look very similar, are initially in a possession of an absent-minded Oxford student Alice. The first qubit is in a precious quantum state and this state is needed urgently for an experiment in Cambridge. Alice's colleague, Bob, pops in to collect the qubit. Once he is gone Alice realises that by mistake she gave him not the first but the third qubit, the one which is entangled with the second qubit (see the figure below).



The situation seems to be hopeless – Alice does not know the quantum state of the first qubit, Bob is now miles away and her communication with him is limited to at most one tweet. However, Alice and Bob are both very clever and attended the "Introduction to Quantum Information Science" course at Oxford. Can Alice rectify her mistake and save Cambridge science?

2.4.B **Partial traces and reduced density operators.** Consider two qubits in the quantum state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|1\rangle \otimes \left(\sqrt{\frac{2}{3}}|0\rangle + \sqrt{\frac{1}{3}}|1\rangle\right) + |0\rangle \otimes \left(\sqrt{\frac{2}{3}}|0\rangle - \sqrt{\frac{1}{3}}|1\rangle\right)\right]. \qquad (1)$$

You obtain reduced density operators by taking partial traces, e.g. the partial trace over $\mathcal{H}_B$ is defined for the tensor product operators,
$\mathrm{tr}_B\,(A \otimes B) = A\,(\mathrm{tr}\,B)$
and extended to any other operator on $\mathcal{H}_A \otimes \mathcal{H}_B$ by linearity. See the Prerequisite Material.

(1) What is the density operator ρ of the two qubits corresponding to state $|\psi\rangle$? Write it in the Dirac notation and explicitly as a matrix in the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$.

(2) Find the reduced density operators ρ_1 and ρ_2 of the first and the second qubit, respectively. Again, write them in the Dirac notation and as matrices in the computational basis.

2.5.B **Trace distance.** The trace norm of a matrix A is defined as

$$\|A\|_{tr} = \mathrm{tr}\left(\sqrt{A^\dagger A}\right).$$

(1) Show that the trace norm of any self-adjoint matrix is the sum of the absolute values of its eigenvalues. What is the trace norm of a density matrix?

(2) The trace distance between density matrices ρ_1 and ρ_2 is defined as

$$d(\rho_1, \rho_2) = \frac{1}{2}\|\rho_1 - \rho_2\|_{tr}.$$

What is the trace distance between two pure states $|\phi\rangle$ and $|\psi\rangle$?

2.6.B **How well can we distinguish two quantum states?.** If a physical system is equally likely to be prepared either in state ρ_1 or state ρ_2 then a single measurement can distinguish between the two preparations with the probability at most

$$\frac{1}{2}\left[1 + d(\rho_1, \rho_2)\right].$$

(1) Suppose ρ_1 and ρ_2 commute. Use the spectral decomposition of ρ_1 and ρ_2 in their common eigenbasis and describe the optimal measurement that can distinguish between the two states. What is the probability of success?

This special case is essentially a classical problem of differentiating between two probability distributions.

(2) Suppose you are given one of the two, randomly selected, qubits of state $|\psi\rangle$ in Eq. (1). What is the maximal probability with which you can determine whether it is the first or the second qubit?

2.7.B **Bloch vectors.** Any density matrix of a single qubit can be parametrised by the three real components of the Bloch vector $\vec{s} = (s_x, s_y, s_z)$ and written as

$$\varrho = \tfrac{1}{2}\left(\mathbb{1} + \vec{s}\cdot\vec{\sigma}\right),$$

where σ_x, σ_y and σ_z are the Pauli matrices, and $\vec{s}\cdot\vec{\sigma} = s_x\sigma_x + s_y\sigma_y + s_z\sigma_z$.

(1) Check that such parametrised ϱ has all the mathematical properties of a density matrix as long as the length of the Bloch vector does not exceed 1.

(2) Draw the Bloch sphere and mark all the convex combinations of states $|0\rangle$ and $|1\rangle$, i.e. the states of the form

$$\rho = p_0 |0\rangle\langle 0| + p_1 |1\rangle\langle 1|,$$

where p_0 and p_1 are non-negative and $p_0 + p_1 = 1$. How would you generate such states?

(3) Draw the Bloch sphere and mark the Pauli eigenstates and all the convex combinations of the Pauli eigenstates.

(4) A qubit in state $|0\rangle$ is modified by a long sequence of randomly selected Clifford gates. You remember the sequence at first, but as time passes you are less and less certain what it was, until you completely forget it. Explain why, from your perspective, the final state of the qubit has a Bloch vector that lies somewhere inside the octahedron with vertices representing the six eigenstates of the Pauli operators X, Y, and Z. Where is this Bloch vector when you still remember the Clifford sequence, and where is it when you have completely forgotten the sequence?

Remember from Question 1.6 that a Clifford gate will map any Pauli eigenstate to another Pauli eigenstate.

(5) Two qubits are in quantum states described by their respective Bloch vectors, $\vec{s}_1$ and $\vec{s}_2$. What is the trace distance between the two quantum states?