INTRODUCTION TO QUANTUM INFORMATION SCIENCE

Problem Sheet 1
Hilary Term

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Questions Label: A - Bookwork B - Standard C - Challenging/Optional

1.1.A **Omnipresent Wolfgang Pauli and his ubiquitous matrices.** The three Pauli matrices $\sigma_1 \equiv \sigma_x \equiv X$, $\sigma_2 \equiv \sigma_y \equiv Y$, and $\sigma_3 \equiv \sigma_z \equiv Z$, here supplemented by the identity matrix $\sigma_0 \equiv \mathbb{1}$, are written in the standard basis $\{|0\rangle, |1\rangle\}$ as

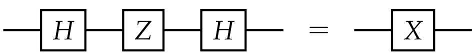
$$\mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

BIT FLIP PHASE FLIP

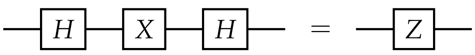
The Pauli matrices are unitary as well as Hermitian. They square to the identity
$$X^2 = Y^2 = Z^2 = \mathbb{1}.$$
They anticommute
$$XY + YX = 0,$$
$$XZ + ZX = 0,$$
$$YZ + ZY = 0,$$
and satisfy
$$XY = iZ,$$
$$YZ = iX,$$
$$ZX = iY.$$
Their trace is zero and their determinant is -1.

(1) Find eigenvalues and eigenvectors of the three Pauli matrices.

(2) The two Pauli gates, X and Z, are often referred to as the bit flip and the phase flip respectively; we will use this terminology later on, when we discuss quantum error correction. Show that the Hadamard gate $H = \frac{1}{\sqrt{2}}(X + Z)$ turns phase flips into bit flips, $HZH = X$,



and bit flips into phase flips $HXH = Z$,



The set of complex $N \times N$ matrices form a Hilbert space with the inner product $(A|B) = \frac{1}{N} \operatorname{Tr} A^\dagger B$. This inner product is often called the *Hilbert-Schmidt product*.

Here vector $\vec{a}$ has components a_x, a_y, a_z and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$.

(3) Given that any 2×2 complex matrix A can be written in the basis of the identity plus the three Pauli matrices as:
$$A = a_0\mathbb{1} + \vec{a} \cdot \vec{\sigma} \equiv a_0\mathbb{1} + a_x\sigma_x + a_y\sigma_y + a_z\sigma_z,$$
show that the coefficients a_k are given by the inner products $a_k = (\sigma_k|A) = \frac{1}{2} \operatorname{Tr} \sigma_k A$. If A is Hermitian then these coefficients are real numbers. Why?

1.2.A **Pauli group.** The three Pauli matrices and the identity form a group under multiplication for when we multiply two Pauli matrices we get another Pauli matrix... well, almost. Explain why the full one-qubit Pauli group $\mathcal{P}_1$ has 16 elements:
$$\pm\mathbb{1}, \pm X, \pm Y, \pm Z, \pm i\mathbb{1}, \pm iX, \pm iY, \pm iZ.$$
Show that $\langle i\mathbb{1}, X, Z\rangle$ is a generating set of this group.

1.3.A **Get stabilized.** We say that a unitary S stabilizes $|\psi\rangle$ if $S|\psi\rangle = |\psi\rangle$.

We will use stabilizers to define vectors and vectors subspaces. Right now it may look like an unnecessary complication, but bear with us...

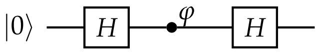
(1) Show that the set of stabilizers of $|\psi\rangle$ forms a group (known as the stabiliser group).

(2) Which states are stabilized by the Pauli matrices X, Y, Z and which by $-X$, $-Y$ and $-Z$? Which states are stabilized by the identity $\mathbb{1}$ and which by $-\mathbb{1}$?

(3) What are the stabilizer groups of the computational basis states, $|0\rangle$ and $|1\rangle$?

1.4.A **The golden circuit.** Here is a single qubit interference represented by the Hadamard – phase shift – Hadamard circuit. What is the role of the first Hadamard gate, the phase shift gate and the second Hadamard gate?

This exercise is important. It really is. Honestly, if you do not understand this circuit you will not get much out of this course.



Step through the execution of this circuit and write down the state of the qubit at each stage of the computation. Comment on a special case of $\varphi = \pi$.

1.5.B **Just Hadamard and Phase.** You are given an unlimited supply of the Hadamard and the phase gates

$$\mathrm{H} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathrm{S} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}.$$

How would you implement $S^\dagger$ and the three Pauli gates?

1.6.B **You know Wolfgang Pauli, now meet William Clifford.** Clifford group on a single qubit, Cl_1, is the group of unitaries generated by the Hadamard and S gates: $Cl_1 = \langle \mathrm{H}, \mathrm{S} \rangle$.

Clifford normalizes Pauli. If this makes no sense to you, please look up centralizers and normalizers.

(1) Show that, under conjugation, Clifford gates $\mathrm{C} \in Cl_1$ map Pauli operators to Pauli operators: $CPC^\dagger = P'$ (modulo phase factors), where P and P' are two Pauli operators. In other words, the Clifford group is defined as the group of unitaries that normalize the Pauli group.

(2) Explain why any circuit composed only of the single qubit Clifford gates maps the set of Pauli eigenstates to the set of Pauli eigenstates.

1.7.B **Tensor products of Pauli operators.** A 2-qubit Pauli operator is a tensor product of any two Pauli operators $(\mathbb{1}, X, Y, Z)$ with pre-factor $+1$ or -1. Using the properties of single-qubit Pauli-operator in **Question 1.1** show that all 2-qubit Pauli operators have the following properties

Given some operators U_i, V_i and some scalars λ_i, we know the following properties of the tensor products of operators:
$(U_1 \otimes U_2)^\dagger = U_1^\dagger \otimes U_2^\dagger$
$\lambda_1 U_1 \otimes \lambda_2 U_2 = (\lambda_1\lambda_2)(U_1 \otimes U_2)$
$(U_1 \otimes U_2)(V_1 \otimes V_2) = U_1V_1 \otimes U_2V_2$

(1) They are both unitary and Hermitian.

(2) They are self-inverse and have eigenvalues ± 1.

(3) Any two operators either commute or anticommute.

Similar arguments can be extended to n-qubit Pauli operator. Now consider the following two 10-qubit Pauli operators

$$X \otimes X \otimes \mathbb{1} \otimes Z \otimes \mathbb{1} \otimes Y \otimes Z \otimes \mathbb{1} \otimes Z \otimes Z$$
$$X \otimes Y \otimes X \otimes \mathbb{1} \otimes \mathbb{1} \otimes Y \otimes X \otimes \mathbb{1} \otimes Z \otimes X$$

(4) Do they commute or anticommute? There is a simple rule that allows to answer this question immediately, without any algebra. Can you see it?

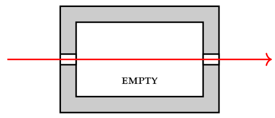
1.8.B **Unitary evolution vs Hamiltonian for independent subsystems.** If subsystem $\mathcal{S}_1$ undergoes a unitary transformation U_1 and subsystem $\mathcal{S}_2$ undergoes a transformation U_2, then the overall unitary evolution is described by the operator $U_1 \otimes U_2$. Now, suppose that both subsystems evolve continuously in time and are characterised by the Hamiltonians H_1 and H_2. What is the overall Hamiltonian?

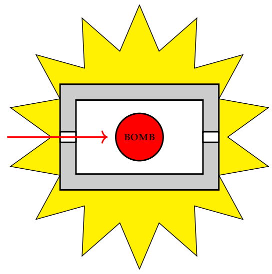
Hint: Over an infinitely small time interval dt, the subsystems evolve by $U_1 = \mathbb{1} - iH_1dt$ and $U_2 = \mathbb{1} - iH_2dt$, respectively.

1.9.B **The Quantum Bomb Tester.** You have been drafted by the government to help in the demining effort in a former war-zone. In particular, retreating forces have left

This is a slightly modified version of a bomb testing problem described by Avshalom Elitzur and Lev Vaidman in *Quantum-mechanical interaction-free measurement*, Found. Phys. **47**, 987-997 (1993).

very sensitive bombs in some of the sealed rooms. The bombs are configured such that if even one photon of light is absorbed by the fuse (i.e. if someone looks into the room), the bomb will go off. Each room has an input and output port which can be hooked up to external devices. An empty room will let light go from the input to the output ports unaffected, whilst a room with a bomb will explode if light is shone into the input port and the bomb absorbs even just one photon.





Your task is to find a way of determining whether a room has a bomb in it without blowing it up, so that specialised (limited and expensive) equipment can be devoted to defusing that particular room. You would like to know with certainty whether a particular room had a bomb in it.

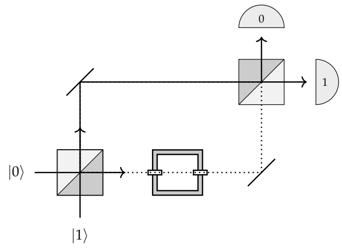
(1) To start with, consider the setup (see the margin) where the input and output ports are hooked up in the lower arm of a Mach-Zehnder interferometer (with symmetric beam splitters).

(a) Assume an empty room. Send a photon to input port $|0\rangle$. Which detector, at the output port, will register the photon?

(b) Now assume that the room does contain a bomb. Again, send a photon to input port $|0\rangle$. Which detector will register the photon and with which probability?

(c) Design a scheme that allows you – at least part of the time – to decide whether a room has a bomb in it without blowing it up. If you iterate the procedure, what is its overall success rate for the detection of a bomb without blowing it up?

Hint: Consider the setup where the input and output ports are hooked up in one of the arms of a Mach-Zehnder interferometer.



(2) Assume that the two beam splitters in the interferometer are different. Say the first beamsplitter reflects incoming light with probability r and transmits with probability $t = 1 - r$ and the second one transmits with probability r and reflects with probability t. Would the new setup improve the overall success rate of the detection of a bomb without blowing it up?

(3) There exists a scheme, involving many beamsplitters and something called "quantum Zeno effect", such that the success rate for detecting a bomb without blowing it up approaches 100%. Try to work it out or find a solution on internet.