SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C7.4
Honour School of Mathematics and Computer Science Part C: Paper C7.4
Honour School of Mathematics and Theoretical Physics Part C: Paper C7.4
Master of Science in Mathematical Sciences: Paper C7.4
Master of Science in Mathematical and Theoretical Physics: Paper C7.4

Introduction to Quantum Information

TRINITY TERM 2024

Friday 31 May, 2:30pm to 4:15pm

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.

You should ensure that you observe the following points:

- start a new answer booklet for each question which you attempt.
- indicate on the front page of the answer booklet which question you have attempted in that booklet.
- cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such booklet and attach these answer booklets at the back of your work.
- hand in your answers in numerical order.

If you do not attempt any questions, you should still hand in an answer booklet with the front sheet completed.

Do not turn this page until you are told that you may do so

1. The two most popular quantum gates are the Hadamard gate and the controlled-NOT gate.

   (a) [2 marks] What are the matrix representations of these two gates in the standard computational basis?

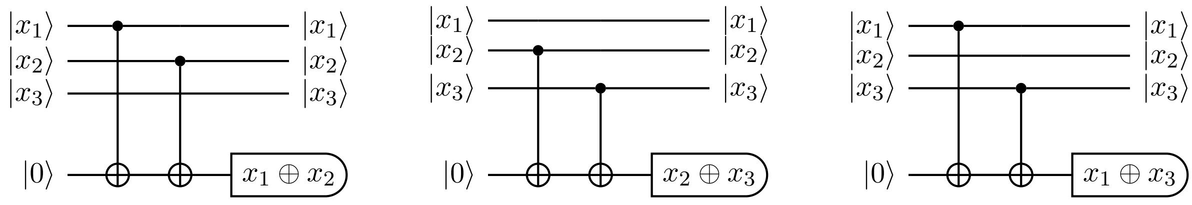
The Hadamard transform on n qubits is defined as

$$|x\rangle \mapsto \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle ,$$

where $x, y \in \{0,1\}^n$ and $x \cdot y \equiv (x_1 \cdot y_1) \oplus \ldots \oplus (x_n \cdot y_n)$.

   (b) [2 marks] Sketch the quantum circuit for implementing a three-qubit Hadamard transform.

The three circuits below implement three different parity checks: $x_1 \oplus x_2$, $x_2 \oplus x_3$ and $x_1 \oplus x_3$,



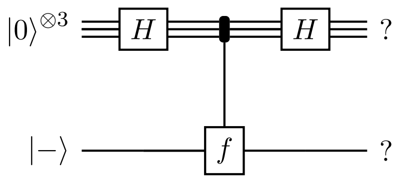
Each circuit can also be viewed as implementing a Boolean function $f\colon \{0,1\}^3 \to \{0,1\}$ of the form

$$f(x) = a \cdot x \equiv (a_1 \cdot x_1) \oplus (a_2 \cdot x_2) \oplus (a_3 \cdot x_3),$$

for some $a \in \{0,1\}^3$.

   (c) [5 marks] What are the binary strings a corresponding to the three parity checks?

One of the three parity-check circuits was chosen uniformly at random and given to you in the form of a black box. Your task is to identify which one of the three was chosen using the black box only once and with zero probability of error. This can be accomplished using the following circuit:



In the diagram, H denotes the Hadamard transform on 3 qubits in the first register, and f represents the oracle operation $f : |x\rangle |y\rangle \mapsto |x\rangle |y \oplus f(x)\rangle$. The single-qubit second register, prepared in the state $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$, receives the value $f(x)$.

   (d) [6 marks] Step through the execution of the circuit, write down quantum states after each computational step, and explain how the value of a is obtained.

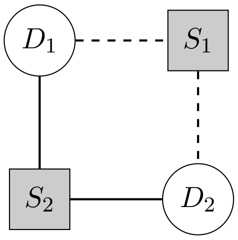
   (e) [4 marks] Suppose the second register is prepared in the state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. What would the output of the circuit be?

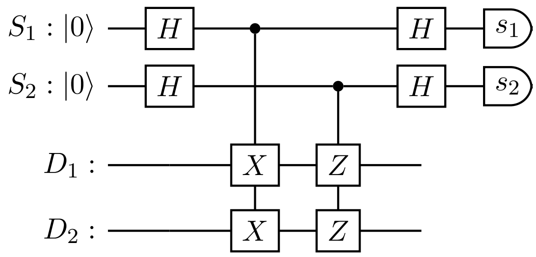
   (f) [6 marks] Now suppose that the input in the second register has decohered to the maximally mixed state. Explain why it is still possible to distinguish the three circuits with zero error probability as long as an inconclusive result is allowed. What is the probability of the inconclusive answer? What is the probability of the inconclusive answer if a is chosen to be not one of the three but one of the eight possible 3-bit binary strings?

2. We define a stabiliser group $\mathcal{S}$ by picking two generators: the parity-check operators X_1X_2 and Z_1Z_2, where the subscript denotes the data qubit on which the Pauli operator acts (e.g. $X_1X_2 = X \otimes X$).

   (a) [2 marks] Specify all the elements of this group $\mathcal{S}$, as well as the state they all stabilise.

Consider the following Tanner graph (left) composed of two data qubits, D_1 and D_2, and two parity-check qubits, S_1 and S_2, which carry the parity-check results (s_1s_2) associated with the stabilisers $S_1 = X_1X_2$ and $S_2 = Z_1Z_2$. In the Tanner graph, solid lines refer to the Z-checks and dashed ones to the X-checks.



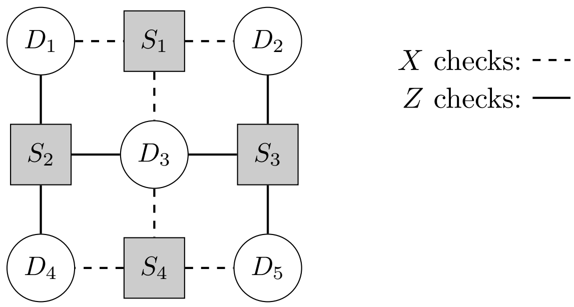


The corresponding parity check circuit is shown on the right. The data qubits, D_1 and D_2, are prepared in the state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, and the parity-check qubits are initially both in state $|0\rangle$.

   (b) [2 marks] Pauli operators either commute or anticommute. Which feature of the Tanner graph above tells us that S_1 and S_2 commute?

   (c) [5 marks] One of the two data qubits experiences a bit flip, a phase flip, or both. Will the measurement of the stabilisers S_1 and S_2 detect the error? Will it identify the affected qubit? Will it be possible to restore the two qubits to the original state? Provide brief explanations for each of your answers.

The Tanner graph below represents a surface code with five data qubits, $D_1, \ldots, D_5$, and four parity check qubits, $S_1, \ldots, S_4$.



   (d) [3 marks] Explain why the set of operators $\{S_1, \ldots, S_4\}$ given in the Tanner graph forms a valid set of stabiliser generators.

   (e) [1 mark] How many logical qubits are encoded with this code?

   (f) [5 marks] Identify a weight-2 X operator (i.e. a tensor product of two physical X operators) that commutes with all elements in the stabiliser group, but is not in the stabiliser group. Do the same for a weight-2 Z operator. Explain why these two operators can be used to represent logical X and Z operators, respectively.

   (g) [3 marks] Can this code detect all single-qubit errors? What is the distance of this code?

   (h) [4 marks] Suppose a single-qubit Pauli error has occurred and the stabiliser measurements give the error syndrome $(s_1s_2s_3s_4) = (1100)$. What kind of error occurred and on which qubit? Answer the same questions for the error syndrome (1000).

3. A quantum channel $\mathcal{A}$ is a completely-positive trace-preserving linear map $\mathcal{A} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}')$ that transforms density operators ρ on $\mathcal{H}$ into density operators $\rho' = \mathcal{A}(\rho)$ on $\mathcal{H}'$. The dimensions of the two Hilbert spaces, $\mathcal{H}$ and $\mathcal{H}'$, may be different. Any quantum channel admits a Kraus decomposition

$$\mathcal{A}(\rho) = \sum_{i=1}^{n} A_i \rho A_i^\dagger,$$

where $A_1, \ldots, A_n$ are the Kraus operators, which satisfy $\sum_{i=1}^{n} A_i^\dagger A_i = \mathbb{1}$.

(a) [2 marks] Show that a composition of two quantum channels with Kraus operators $\{A_i\}$ and $\{B_j\}$ (respectively) is another quantum channel.

The Kraus decomposition is not unique. All possible sets of Kraus operators associated with a given quantum channel are unitarily related.

(b) [4 marks] Given a set of Kraus operators $\{A_j\}$ of size n and some unitary $m \times m$ matrix, where $m \geqslant n$, show that the set of operators $\{A_i'\}$ of size m, where $A_i' = \sum_{j=1}^{n} u_{ij} A_j$, is another set of Kraus operators that represents the same quantum channel.

(c) [3 marks] Explain why the identity channel $\rho' = \mathbb{1}\rho\mathbb{1} = \sum_i A_i \rho A_i^\dagger$ can only have Kraus operators that are proportional to the identity: $A_i = \lambda_i \mathbb{1}$, for some complex number λ_i. Show that they must further satisfy $\sum_i |\lambda_i|^2 = 1$.

A quantum channel $\mathcal{A}$ is said to be reversible if there exists a quantum channel $\mathcal{R}$ (called the recovery channel) such that $(\mathcal{R} \cdot \mathcal{A})(\rho) = \rho$ for any density operator $\rho \in \mathcal{B}(\mathcal{H})$.

(d) [4 marks] Let $\mathcal{R}$ be represented by the set of Kraus operators $\{R_i\}$. Explain why $R_i A_j = \lambda_{ij} \mathbb{1}$ for some complex number λ_{ij}. Show that, if a quantum channel $\mathcal{A}$ is reversible, then its Kraus operators satisfy $A_j^\dagger A_i = \sigma_{ji} \mathbb{1}$, where $\sigma_{ji} = \sum_k \lambda_{kj}^\star \lambda_{ki}$.

(e) [3 marks] Show that, when switching to another Kraus representation of the same channel, the reversibility criterion above still holds but with a different σ_{ji}.

(f) [2 marks] Show that σ_{ji}, viewed as a matrix, is a density operator.

An isometry is a linear map $V : \mathcal{H} \to \mathcal{H}'$ such that $V^\dagger V = \mathbb{1}$, where the identity operator acts on $\mathcal{H}$. Consider a random isometric channel $\mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}')$,

$$\rho \mapsto \sum_i p_i V_i \rho V_i^\dagger,$$

in which isometry V_i is chosen with probability p_i, and any two isometries V_i and V_j are mutually orthogonal, $V_i^\dagger V_j = \delta_{ij} \mathbb{1}$, i.e. the images of $\mathcal{H}$ under V_i and V_j are orthogonal subspaces.

(g) [4 marks] Describe a measurement on $\mathcal{H}'$ from which you can learn which particular isometry V_i was selected? How can you reverse the action of this channel?

(h) [3 marks] Explain why it is possible to reverse certain random isometric channels but not random unitary channels, except the trivial case of a single unitary operation.