SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C7.4

Honour School of Mathematics and Statistics Part C: Paper C7.4

Honour School of Mathematical and Theoretical Physics Part C: Paper C7.4

Master of Science in Mathematical Sciences: Paper C7.4

Master of Science in Mathematical and Theoretical Physics: Paper C7.4

INTRODUCTION TO QUANTUM INFORMATION

TRINITY TERM 2021

Monday 07 June

Opening time: 09:30 (BST)

Mode of completion: Handwritten

You have 1 hour 45 minutes writing time to complete the paper and up to 30 minutes technical time to upload your answer file.

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.

You should ensure that you observe the following points:

1. Write with a black or blue pen OR with a stylus on tablet (colour set to black or blue).
2. On the first page, write
   - your candidate number
   - the paper code
   - the paper title
   - and your course title (e.g. FHS Mathematics and Statistics Part C)
   - but ***do not*** enter your name or college.
3. For each question you attempt,
   - start writing on a new sheet of paper,
   - indicate the question number clearly at the top of each sheet of paper,
   - number each page
4. Before scanning and submitting your work,
   - on the first page, in numerical order, write the question numbers attempted,
   - cross out all rough working and any working you do not want to be marked,
   - and orient all scanned pages in the same way.
5. Submit all your answers to this paper as a ***single PDF*** document

If you do not attempt any questions at all on this paper, you should still submit a single page indicating that you have opened the exam but not attempted any questions. Please make sure to write your candidate number on this single page.

1. Any density matrix of a single qubit ρ can be parameterised by the three real components of the Bloch vector $\vec{s} = (s_x, s_y, s_z)$,

$$\rho = \frac{1}{2}\left(\mathbb{1} + s_x X + s_y Y + s_z Z\right),$$

where X, Y and Z are the Pauli operators.

(a) [3 marks] Explain why the Bloch vectors form a unit ball in $\mathbb{R}^3$. Draw the Bloch sphere and mark all the convex combinations of states $|0\rangle$ and $|1\rangle$, i.e. the states of the form

$$\rho = p_0 |0\rangle\langle 0| + p_1 |1\rangle\langle 1|,$$

where p_0 and p_1 are non-negative and $p_0 + p_1 = 1$. How would you generate such states?

(b) [4 marks] Draw the Bloch sphere and mark the Pauli eigenstates and all the convex combinations of the Pauli eigenstates.

(c) [3 marks] Gates X, Y, Z rotate the Bloch sphere by π about the x, y and z axes, respectively. Use the Bloch sphere language to describe the action of the Hadamard gate H, the phase gate S, and the T gate,

$$\mathrm{H} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathrm{S} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad \mathrm{T} = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix}.$$

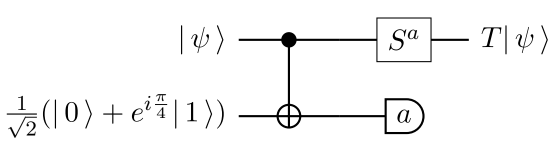
The *Clifford group on a single qubit*, Cl_1, is the group of unitaries generated by the Hadamard and S gates: $Cl_1 = \langle \mathrm{H}, \mathrm{S} \rangle$.

(d) [4 marks] Show that, under conjugation, Clifford gates $\mathrm{C} \in Cl_1$ map Pauli operators to Pauli operators: $CPC^\dagger = P'$ (modulo phase factors), where P and P' are two Pauli operators. Explain why any circuit composed only of the single qubit Clifford gates maps the set of Pauli eigenstates to the set of Pauli eigenstates.

(e) [5 marks] A qubit in state $|0\rangle$ is modified by a long sequence of randomly selected Clifford gates. You remember the sequence at first, but as time passes you are less and less certain what it was, until you completely forget it. Explain why, from your perspective, the final state of the qubit has a Bloch vector that lies somewhere inside the octahedron with vertices representing the six eigenstates of the Pauli operators X, Y, and Z. Where is this Bloch vector when you still remember the Clifford sequence, and where is it when you have completely forgotten the sequence?

The Clifford group on n qubits Cl_n is generated by the gates H, S and C-NOT.

(f) [6 marks] Suppose we can perform only Clifford gates, measurements in the standard basis, and prepare the quantum state $\frac{1}{\sqrt{2}}(|0\rangle + e^{i\frac{\pi}{4}}|1\rangle)$ (by some means). Then gate T acting on any state $|\psi\rangle$ can be implemented using the following circuit



where the second qubit is measured in the standard basis, with the outcome $a = 0, 1$ registered, and then the gate S^a is applied to the first qubit. Step through the execution of this circuit and explain how the T gate is implemented. Why is it important to supplement Clifford gates with the T gate?

2. A source repeatedly generates two entangled qubits in the state $|\,\Omega\,\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$,

$$|\,\Omega\,\rangle = \frac{1}{\sqrt{2}}(|\,0\,\rangle|\,0\,\rangle + |\,1\,\rangle|\,1\,\rangle).$$

One qubit is sent to Alice and one to Bob.

(a) [3 marks] When Alice measures her qubit in the standard Z basis, she *instantaneously* knows whether Bob, who may be miles away, will observe outcome 0 or 1 when he measures his qubit in the Z basis. Explain why these correlations cannot be used for instantaneous communication but they can be used for generating cryptographic keys.

(b) [4 marks] Show that for any two operators $A, B \in \mathcal{B}(\mathbb{C}^2)$, we have

$$\langle\,\Omega\,|A \otimes B|\,\Omega\,\rangle = \frac{1}{2}\,\mathrm{tr}\,A^T B.$$

(c) [4 marks] Let A and B be two observables measured by Alice and Bob, respectively,

$$A = \cos\alpha Z + \sin\alpha X, \quad \text{and} \quad B = \cos\beta Z + \sin\beta X, \tag{1}$$

where X and Z are the Pauli operators. Show that

$$\langle\,\Omega\,|A \otimes B|\,\Omega\,\rangle = \cos(\alpha - \beta).$$

What is the probability that the results registered by Alice and Bob upon measuring these observables are identical?

Let A_1, A_2, B_1 and B_2 be the observables defined by using the angles $\alpha_1 = \frac{\pi}{2}$, $\alpha_2 = 0$, $\beta_1 = \frac{\pi}{4}$ and $\beta_2 = \frac{3\pi}{4}$ (respectively) in Equation (1). Alice and Bob perform a statistical test (known as the CHSH test) in which Alice repeatedly measures either A_1 or A_2, and Bob either B_1 or B_2. For each run they choose, randomly and independently from each other, which observable to measure, and then check whether the following conditions are satisfied:

$$A_1 = B_1, \quad A_1 = B_2, \quad A_2 = B_1, \quad A_2 \neq B_2. \tag{2}$$

In each run they are able to check only *one* of the four conditions depending on the pair of observables they choose to measure.

(d) [3 marks] Show that their probability P_s of success (i.e. the asymptotic fraction of runs in which they find that the outcomes agree with the conditions in (2)) is given by $P_s = \cos^2\frac{\pi}{8}$.

The CHSH test can be performed using any two devices, $\mathcal{A}$ and $\mathcal{B}$, that are "black boxes" of unknown design, as long as each device has two adjustable settings, say A_1, A_2 for $\mathcal{A}$ and B_1, B_2 for $\mathcal{B}$, and each can be run to generate outcome ± 1. The probability of success in any such CHSH test cannot exceed $P_s = \cos^2\frac{\pi}{8}$. The maximum value is achieved only when the settings correspond to the measurements on qubits in state $|\,\Omega\,\rangle$, as described above (modulo some simple relabelling). The test is rigid — there is no other way to maximise the probability of success.

(e) [6 marks] An adversary, Eve, who manufactured, pre-programmed, and sold the two devices to Alice and Bob, claims that for each run of the CHSH test she knows the outcomes A_1, A_2, B_1, and B_2 with certainty. Assume that Eve assigned numerical values ± 1 to A_1, A_2, B_1, and B_2 so that the outcomes are predetermined. Alice and Bob run the CHSH test using their unreliable devices and obtain $P_s \approx 0.85$. What should they conclude?

(f) [5 marks] Would the CHSH test be conclusive if Alice and Bob, in their "random" choices of measurements, relied on random number generators supplied by their adversary Eve?

3. Any state vector $|\psi\rangle_{AB}$ in $\mathcal{H}_A \otimes \mathcal{H}_B$ can be expressed in a standard form, known as the Schmidt decomposition:

$$|\psi\rangle_{AB} = \sum_{k=1}^{d} \sqrt{p_k}\, |a_k\rangle \otimes |b_k\rangle, \tag{3}$$

where $d = \min(\dim \mathcal{H}_A, \dim \mathcal{H}_B)$, the Schmidt coefficients $p_k \geqslant 0$ are uniquely determined, and $\{|a_i\rangle\}$ and $\{|b_j\rangle\}$ are orthonormal bases in $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively.

(a) [3 marks] Given the Schmidt decomposition in Equation (3), what are the reduced density operators ρ_A and ρ_B associated with the subsystems A and B?

(b) [3 marks] Alice and Bob receive qubits from an external source controlled by Eve, who claims that they are the two qubits from a two-qubit entangled pure state. After many runs, Alice and Bob estimate the reduced density operators of their qubits and find them to be diagonal in the same basis, with

$$\rho_A = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.1 \end{pmatrix} \qquad \text{and} \qquad \rho_B = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.7 \end{pmatrix}. \tag{4}$$

Is Eve telling the truth?

(c) [10 marks] For any given real number r, we define the map $\mathcal{E}_r : \mathcal{B}(\mathbb{C}^2) \to \mathcal{B}(\mathbb{C}^2)$ by

$$\mathcal{E}_r(\mathbb{1}) = \mathbb{1}, \quad \mathcal{E}_r(X) = rX, \quad \mathcal{E}_r(Y) = rY, \quad \mathcal{E}_r(Z) = rZ,$$

where X, Y, and Z are the Pauli operators.

(i) Describe the action of this map on density operators ρ in terms of the Bloch vector parametrisation $\vec{s} = (s_x, s_y, s_z)$,

$$\rho = \frac{1}{2}\left(\mathbb{1} + s_x X + s_y Y + s_z Z\right).$$

(ii) For what range of values of r is the map $\mathcal{E}_r$ positive?

(iii) Write down the Choi matrix for $\mathcal{E}_r$.

(iv) For what range of values of r is the map $\mathcal{E}_r$ completely positive?

(d) [5 marks] Suppose Eve is able to implement $\mathcal{E}_r$ and any other unitary operation on a qubit. She does not know the basis in which both ρ_A and ρ_B, in Equation (4), are diagonal. Can she reliably transform either ρ_A to ρ_B, or ρ_B to ρ_A?

(e) [4 marks] Would the answer be any different if she knew the basis in which the two density operators are diagonal?