A14687W1

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C7.4
Honour School of Mathematics and Philosophy Part C: Paper C7.4
Honour School of Mathematics and Computer Science Part C: Paper C7.4
Honour School of Mathematical and Theoretical Physics Part C: Paper C7.4
Master of Science in Mathematical Sciences: Paper C7.4
Master of Science in Mathematical and Theoretical Physics: Paper C7.4

INTRODUCTION TO QUANTUM INFORMATION

TRINITY TERM 2020

Tuesday 09 June

Opening time: 14:30 (BST)

You have 2 hours 45 minutes to complete the paper and upload your answer file

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.

You should ensure that you observe the following points:

1. Write with a black or blue pen OR with a stylus on tablet (colour set to black or blue).
2. On the first page, write
   - your candidate number
   - the paper code
   - the paper title
   - and your course title (e.g. FHS Mathematics and Statistics Part B)
   - but ***do not*** enter your name or college.
3. For each question you attempt,
   - start writing on a new sheet of paper,
   - indicate the question number clearly at the top of each sheet of paper,
   - number each page
4. Before scanning and submitting your work,
   - add to the first page, in numerical order, the question numbers attempted,
   - cross out all rough working and any working you do not want to be marked,
   - and orient all scanned pages in the same way.
5. Submit a single PDF document with your answers for this paper.

If you do not attempt any questions at all on this paper, you should still submit a single page indicating that you have opened the exam but not attempted any questions. Please make sure to write your candidate number on this single page.

1. The Hadamard transform on n qubits is defined as

$$|\,x\,\rangle \mapsto \frac{1}{\sqrt{2^n}} \sum_{y\in\{0,1\}^n} (-1)^{x\cdot y} |\,y\,\rangle,$$

where $x, y \in \{0,1\}^n$ and $x \cdot y \equiv (x_1 \cdot y_1) \oplus \ldots \oplus (x_n \cdot y_n)$.

(a) [3 marks] Sketch the quantum circuit which effects the Hadamard transform and explain why it is useful as the first operation in quantum algorithms.

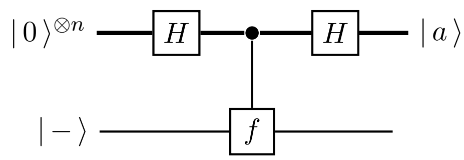
You are presented with an oracle that computes some unknown function $f\colon \{0,1\}^n \to \{0,1\}$, but you are promised that f is of the form

$$f(x) = a \cdot x \equiv (a_1 \cdot x_1) \oplus \ldots \oplus (a_n \cdot x_n),$$

for some fixed $a \in \{0,1\}^n$. Your task is to determine the value of the n-bit string a using the fewest queries possible.

(b) [3 marks] How many calls to the oracle are required to determine a if the oracle is classical?

(c) [7 marks] The circuit below implements a quantum algorithm which outputs the value of a with a single call to the (quantum) oracle. In the diagram the H operations in the first register denote the Hadamard transform on n qubits and the f operation represents the oracle, i.e., a quantum evaluation of $f\colon |\,x\,\rangle|\,y\,\rangle \mapsto |\,x\,\rangle|\,y \oplus f(x)\,\rangle$. The second register, to which the value $f(x)$ is added, contains one qubit prepared in state $|\,-\,\rangle = \frac{1}{\sqrt{2}}(|\,0\,\rangle - |\,1\,\rangle)$.



Step through the execution of this circuit, writing down quantum states of the two registers after each computational step. Explain how the value of a is obtained.

(d) [5 marks] If the state of the second register, $|\,-\,\rangle$, is replaced with $|\,+\,\rangle = \frac{1}{\sqrt{2}}(|\,0\,\rangle + |\,1\,\rangle)$, prove that you learn nothing about the value of a by running this circuit.

(e) [7 marks] The network construction presented here can be generalised to the case of a Boolean function $f : \{0,1\}^n \mapsto \{0,1\}^m$. Suppose the second register contains m qubits and the oracle evaluates the function $f(x) = A \cdot x$ (modulo 2) where A is an $m \times n$ binary matrix. By running the network m times with suitable choices for the states of the second register all the entries of A can be determined. Explain how.

2. Any density matrix of a single qubit can be parameterised by the three real components of the Bloch vector $\vec{s} = (s_x, s_y, s_z)$ and written as

$$\varrho = \tfrac{1}{2}\left(\mathbb{1} + \vec{s}\cdot\vec{\sigma}\right),$$

where σ_x, σ_y and σ_z are the Pauli matrices, and $\vec{s}\cdot\vec{\sigma} = s_x\sigma_x + s_y\sigma_y + s_z\sigma_z$.

(a) [2 marks] Express the eigenvalues of ϱ in terms of the length of $\vec{s}$ and explain why the length of the Bloch vector cannot exceed 1.

(b) [3 marks] Show that for any two density operators ϱ_1 and ϱ_2, $\mathrm{Tr}(\varrho_1\varrho_2) = {}^1\!/_2(1 + \vec{s}_1\cdot\vec{s}_2)$, where $\vec{s}_1$ and $\vec{s}_2$ are the Bloch vectors of ϱ_1 and ϱ_2, respectively.

(c) [3 marks] Show that unitary evolutions, $\varrho \mapsto U\varrho U^\dagger$ preserve the scalar product of Bloch vectors and deduce that such evolutions correspond to rotations of the Bloch sphere. Describe these rotations for the Pauli unitaries σ_x, σ_y and σ_z.

(d) [3 marks] A qubit in state ϱ is transmitted through a depolarising channel that effects a completely positive map

$$\varrho \mapsto (1-p)\varrho + \frac{p}{3}\left(\sigma_x\varrho\sigma_x + \sigma_y\varrho\sigma_y + \sigma_z\varrho\sigma_z\right),$$

for some $0 \leqslant p \leqslant 1$. Show that under this map the Bloch vector associated with ϱ shrinks by the factor $(3-4p)/3$.

The trace norm of a matrix A is defined as

$$|A|_{\mathrm{tr}} = \mathrm{Tr}\left(\sqrt{A^\dagger A}\right).$$

(e) [2 marks] Explain why the trace norm of any self-adjoint matrix is the sum of the absolute values of its eigenvalues. What is the trace norm of a density matrix?

If a physical system is equally likely to be prepared either in state ϱ_1 or state ϱ_2 then a single measurement can distinguish between the two preparations with the probability at most

$$P_S = \frac{1}{2} + \frac{1}{4}\left|\varrho_1 - \varrho_2\right|_{\mathrm{tr}}, \qquad (1)$$

where $\frac{1}{2}\left|\varrho_1 - \varrho_2\right|_{\mathrm{tr}}$ is known as the trace distance between ϱ_1 and ϱ_2.

(f) [4 marks] Explain why the statement above implies that all physically admissible operations can only reduce the trace distance between density operators.

(g) [4 marks] A qubit is equally likely to be prepared either in state ϱ_1 or state ϱ_2. Show that

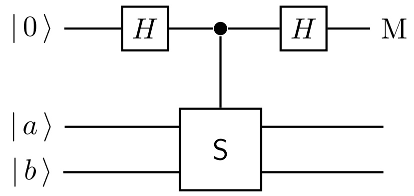
$$P_S = \frac{1}{2} + \frac{1}{4}\left|\vec{s}_1 - \vec{s}_2\right|.$$

(h) [4 marks] Show that unitary evolutions do not degrade distinguishability of quantum states but the depolarising channel does. By how much is P_S decreased by the action of the depolarising channel?

3. The swap gate S on two qubits is defined first on product vectors, $S : |\,a\,\rangle \otimes |\,b\,\rangle \mapsto |\,b\,\rangle \otimes |\,a\,\rangle$ and then extended to sums of product vectors by linearity.

   (a) [3 marks] Show that $P_\pm = \frac{1}{2}(\mathbb{1} \pm S)$ are two orthogonal projectors which form the decomposition of the identity and project on the symmetric and the antisymmetric subspaces. Decompose the state vector $|\,a\,\rangle|\,b\,\rangle$ of two qubits into symmetric and antisymmetric components.

Consider the following "swap-test" quantum circuit composed of two Hadamard gates, one controlled-S operation and the measurement M in the computational basis,



The state vectors $|\,a\,\rangle$ and $|\,b\,\rangle$ of the target qubits are normalised but not orthogonal to each other.

   (b) [5 marks] Step through the execution of this circuit, writing down quantum states of the three qubits after each computational step. What are the probabilities of observing 0 or 1 when the measurement M is performed? Explain why this circuit implements projections on the symmetric and the antisymmetric subspaces of the two target qubits.

   (c) [3 marks] Does the measurement result $M = 0$ imply that $|\,a\,\rangle$ and $|\,b\,\rangle$ are identical? Does the measurement result $M = 1$ imply that $|\,a\,\rangle$ and $|\,b\,\rangle$ are not identical?

   (d) [5 marks] Suppose an efficient quantum algorithm encodes information about a complicated graph into a pure state of a qubit. Graphs which are isomorphic are mapped into the same state of the qubit. Given two complicated graphs your task is to check if they are isomorphic. You can run the algorithm as many times as you want and you can use the "swap-test" circuit. How would you accomplish this task?

   (e) [6 marks] Instead of the state $|\,a\,\rangle \otimes |\,b\,\rangle$ the two target qubits are prepared in some mixed state $\rho_a \otimes \rho_b$. Show that the probability of successful projection of this state on the symmetric subspace is

$$\frac{1}{2}\left(1 + \mathrm{Tr}\,\rho_a \rho_b\right).$$

   (f) [3 marks] Does the measurement result $M = 1$ imply that ρ_a and ρ_b are not identical?

[*In the computational basis, the Pauli matrices* σ_x, σ_y, *and* σ_z *are*

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

They anticommute and square to the identity: $\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = \mathbb{1}$. *They also satisfy:*

$$(\vec{a} \cdot \vec{\sigma})(\vec{b} \cdot \vec{\sigma}) = (\vec{a} \cdot \vec{b})\,\mathbb{1} + i(\vec{a} \times \vec{b}) \cdot \vec{\sigma},$$

for any two Euclidean vectors $\vec{a}$ *and* $\vec{b}$.]