A14687W1

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C7.4
Honour School of Mathematics and Computer Science Part C: Paper C7.4
Honour School of Mathematics and Statistics Part C: Paper C7.4
Honour School of Mathematical and Theoretical Physics Part C: Paper C7.4
Master of Science in Mathematical and Theoretical Physics: Paper C7.4
Master of Science in Mathematical Sciences: Paper C7.4

INTRODUCTION TO QUANTUM INFORMATION

Trinity Term 2019

MONDAY, 10 JUNE 2019, 9.30am to 11.15am

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.

You should ensure that you:

- *start a new answer booklet for each question which you attempt.*
- *indicate on the front page of the answer booklet which question you have attempted in that booklet.*
- *cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such booklet and attach these answer booklets at the back of your work.*
- *hand in your answers in numerical order.*

If you do not attempt any questions, you should still hand in an answer booklet with the front sheet completed.

Do not turn this page until you are told that you may do so

1. The CHSH inequality involves four observables: A_1, A_2, pertaining to qubit $\mathcal{A}$, and B_1, B_2, pertaining to qubit $\mathcal{B}$. Each observable takes values ± 1.

   (a) [4 marks] Consider the local hidden-variable scenario, that is, assume that values ± 1 can be assigned simultaneously to all four observables. The CHSH quantity S is defined as

$$S = A_1(B_1 - B_2) + A_2(B_1 + B_2).$$

   Explain why any statistical average of S must satisfy $-2 \leqslant \langle S \rangle \leqslant 2$, so that classical correlations are bounded by the CHSH inequality,

$$|\langle A_1 B_1 \rangle - \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle + \langle A_2 B_2 \rangle| \leqslant 2.$$

In quantum theory the observables A_1, A_2, B_1, B_2 become 2×2 Hermitian matrices with two eigenvalues ± 1, and $\langle S \rangle$ becomes the expectation value of the 4×4 CHSH matrix

$$S = A_1 \otimes (B_1 - B_2) + A_2 \otimes (B_1 + B_2).$$

   (b) [3 marks] Explain why for any state $|\,\psi\,\rangle$ the expectation value $\langle S \rangle = \langle\,\psi\,|S|\,\psi\,\rangle$ cannot exceed the largest eigenvalue of S.

The largest eigenvalue (in absolute value) of a Hermitian matrix M, denoted by $||M||$, is a matrix norm and has the following properties (which you may use):

$$||M \otimes N|| = ||M||\,||N||, \quad ||MN|| \leqslant ||M||\,||N||, \quad ||M + N|| \leqslant ||M|| + ||N||.$$

   (c) [2 marks] Explain why $||A_k|| = 1$ and $||B_l|| = 1$ $(k, l = 1, 2)$, and show that $||S|| \leqslant 4$.

   One can, however, derive a tighter bound as follows

   (d) [5 marks] Show that

$$S^2 = 4\,\mathbb{1} \otimes \mathbb{1} + [A_1, A_2] \otimes [B_1, B_2].$$

   (e) [3 marks] Explain why the norm of each of the commutators, $||[A_1, A_2]||$ and $||[B_1, B_2]||$, cannot exceed 2 and why $||S^2|| = ||S||^2$.

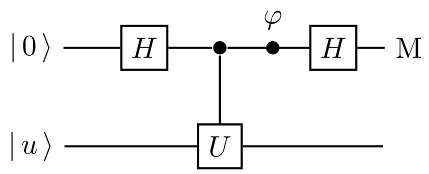
   (f) [4 marks] Show that quantum correlations are bounded by the Tsirelson inequality

$$||S|| \leqslant 2\sqrt{2},$$

   and explain why this implies that $|\langle S \rangle| \leqslant 2\sqrt{2}$.

   (g) [4 marks] How is this inequality modified when the operators A_1, A_2, B_1, B_2 commute with each other? Provide a physical interpretation.

2. Consider the following quantum network composed of the two Hadamard gates, one phase gate φ , one controlled-U operation and the measurement M in the standard basis,



The top horizontal line represents a qubit and the bottom one an auxiliary physical system.

(a) [5 marks] Suppose $|\,u\,\rangle$ is an eigenvector of U, such that $U|\,u\,\rangle = e^{i\alpha}|\,u\,\rangle$. Step through the execution of this network, writing down quantum states of the qubit and the auxiliary system after each computational step. In measurement M outcome 0 is registered with probability Pr(0). What is this probability?

(b) [4 marks] Show that for any pure state $|\,u\,\rangle$ the probability Pr(0) can be expressed as

$$\Pr(0) = \frac{1}{2}\left[1 + \mathrm{Re}\left(\, e^{i\varphi}\langle\, u\,|U|\, u\,\rangle\right)\right].$$

(c) [6 marks] Suppose the auxiliary system is prepared in a mixed state described by the density operator ρ,

$$\rho = p_1|\,u_1\,\rangle\langle\,u_1\,| + p_2|\,u_2\,\rangle\langle\,u_2\,| + \ldots\ldots + p_n|\,u_n\,\rangle\langle\,u_n\,|,$$

where vectors $|\,u_k\,\rangle$ form an orthonormal basis, $p_k \geqslant 0$ and $\sum_{k=1}^{n} p_k = 1$. Show that

$$\Pr(0) = \frac{1}{2}\left[1 + \mathrm{Re}\left(\, e^{i\varphi}\,\mathrm{Tr}(U\rho)\right)\right]. \tag{1}$$

(d) [5 marks] Suppose you have control over the phase gate φ and you can prepare any state of the auxiliary system ρ, and run the circuit as many times as you wish. How would you estimate the trace of U?

(e) [5 marks] Assume the auxiliary system is maximally entangled with some other system that does not participate in the quantum evolution induced by the circuit. The joint state of the two systems is

$$|\,\psi\,\rangle = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}|\,u_i\,\rangle|\,v_i\,\rangle,$$

where vectors $|\,u_i\,\rangle$ and $|\,v_i\,\rangle$ form orthonormal bases in their respective Hilbert spaces. What is the probability Pr(0) is this case? How is it related to the probability given in (1)?

3. Any linear map A acting on density matrices of a qubit can be completely characterised by its action on the four basis matrices $|\,i\,\rangle\langle\,j\,|$, where $i, j = 0, 1$, and represented as a 4×4 matrix, known as the Choi matrix,

$$\tilde{A} = \frac{1}{2}\left[\begin{array}{c|c} A(|\,0\,\rangle\langle\,0\,|) & A(|\,0\,\rangle\langle\,1\,|) \\ \hline A(|\,1\,\rangle\langle\,0\,|) & A(|\,1\,\rangle\langle\,1\,|) \end{array}\right]. \tag{2}$$

Depending on the map A, the Choi matrix may or may not be a density matrix.

(a) [3 marks] Explain why a matrix must be both positive semi-definite and have trace 1 in order to be considered a density matrix.

For a physically admissible map A we require that both A and its extension to any other physical systems, written as $\mathbb{1} \otimes A$, map density operators into density operators. Such maps are called completely positive trace-preserving (CPTP) maps.

(b) [6 marks] Show that the Choi matrix (2) can be expressed as

$$(\mathbb{1} \otimes A)|\,\psi\,\rangle\langle\,\psi\,|, \tag{3}$$

where $|\,\psi\,\rangle = \frac{1}{\sqrt{2}}(|\,00\,\rangle + |\,11\,\rangle)$. What does this tell you about the Choi matrix associated with completely positive maps?

(c) [5 marks] The transpose operation T is defined as $|\,i\,\rangle\langle\,j\,| \to |\,j\,\rangle\langle\,i\,|$. Is the transpose of a density matrix also a density matrix? Using the Choi matrix, as expressed in (3), or otherwise, show that transpose is not a physically admissible operation.

A quantum state of two qubits described by the density matrix ϱ is called separable if ϱ is of the form

$$\varrho = \sum_k p_k\, \rho_k \otimes \upsilon_k,$$

where $p_k \geqslant 0$ and $\sum_{k=1} p_k = 1$. Otherwise ϱ is called entangled.

(d) [5 marks] Show that partial transpose, $\mathbb{1} \otimes T$, maps separable states into separable states.

(e) [6 marks] Consider a quantum state of two qubits described by the density matrix

$$\rho = p|\,\psi\,\rangle\langle\,\psi\,| + (1-p)\frac{1}{4}\mathbb{1} \otimes \mathbb{1}, \quad p \in [0, 1].$$

Apply partial transpose $\mathbb{1} \otimes T$ to this state and check if the resulting matrix is a density matrix. For which values of p does the density matrix ρ represent an entangled state?