A14687W1

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C7.4
Honour School of Mathematical and Theoretical Physics Part C: Paper C7.4
Master of Science in Mathematical and Theoretical Physics: Paper C7.4

INTRODUCTION TO QUANTUM INFORMATION

Trinity Term 2018

SATURDAY, 2 JUNE 2018, 2.30pm to 4.15pm

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.

You should ensure that you:

- *start a new answer booklet for each question which you attempt.*
- *indicate on the front page of the answer booklet which question you have attempted in that booklet.*
- *cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such booklet and attach these answer booklets at the back of your work.*
- *hand in your answers in numerical order.*

If you do not attempt any questions, you should still hand in an answer booklet with the front sheet completed.

Do not turn this page until you are told that you may do so

1. (a) [3 marks] Consider a Boolean function $f : \{0,1\}^n \to \{0,1\}$ and its quantum evaluation,

$$|\, x \,\rangle |\, y \,\rangle \mapsto |\, x \,\rangle |\, y \oplus f(x) \,\rangle,$$

where $x \in \{0,1\}^n$ and $y \in \{0,1\}$ are the binary strings placed in the first and the second register respectively. Explain how the quantum evaluation of f can be reduced to the unitary operation U_f on the first register,

$$U_f |\, x \,\rangle = (-1)^{f(x)} |\, x \,\rangle.$$

What is the geometric interpretation of this transformation?

(b) [2 marks] The reflection in the subspace that is orthogonal to $|\, a \,\rangle$ can be written as

$$V_a = \mathbb{1} - 2|\, a \,\rangle\langle\, a \,|.$$

Provide the geometric interpretation of $UV_aU^\dagger$, where U is a unitary operator.

(c) [5 marks] Let $f(x) = 1$ for $x = s$ and $f(x) = 0$ otherwise. We denote the Hadamard transform on n qubits by H_n and the reflection in the subspace that is orthogonal to $|\, 0 \,\rangle$ by V_0. (The vector $|\, 0 \,\rangle$ represents the binary string of n zeros, $|\, 0 \ldots 0 \,\rangle$.) Describe the action of the Grover iteration operator

$$G = -H_n V_0 H_n U_f,$$

in the plane spanned by $H_n |\, 0 \,\rangle$ and the unknown $|\, s \,\rangle$.

A quantum algorithm $\mathcal{A}$, which solves a certain problem in the complexity class NP, can be viewed as a unitary operation A on n qubits. The result of $A|\, 0 \,\rangle$ is the state $|\, \psi \,\rangle$, which is a superposition of binary strings representing possible, not necessarily correct, outputs. It is known that a subsequent measurement in the computational basis provides a correct answer to the problem with the probability $p = \sin^2\theta \ll 1$ and that $|\, \psi \,\rangle$ can be written as

$$|\, \psi \,\rangle = \sin\theta \,|\, \psi_g \,\rangle + \cos\theta \,|\, \psi_b \,\rangle,$$

where $|\, \psi_g \,\rangle$ and $|\, \psi_b \,\rangle$ are normalised projections of $|\, \psi \,\rangle$ onto the subspace spanned by the binary strings corresponding to good and bad answers, respectively. Let U_f correspond to the Boolean function $f : \{0,1\}^n \to \{0,1\}$ that verifies the outputs of $\mathcal{A}$, that is, $f(x) = 1$ if x is the correct output and $f(x) = 0$ otherwise. You can assume that A, $A^\dagger$ and U_f can be efficiently implemented.

(d) [8 marks] Show that the subspace spanned by $|\, \psi_g \,\rangle$ and $|\, \psi_b \,\rangle$ is invariant under the action of the modified Grover iteration operator Q,

$$Q = -AV_0A^\dagger U_f,$$

and express $Q|\, \psi_g \,\rangle$ and $Q|\, \psi_b \,\rangle$ as linear superpositions of $|\, \psi_g \,\rangle$ and $|\, \psi_b \,\rangle$.

(e) [3 marks] Show that after r applications of Q to the state $|\, \psi \,\rangle$ we obtain

$$Q^r |\, \psi \,\rangle = \sin((2r+1)\theta) |\, \psi_g \,\rangle + \cos((2r+1)\theta) |\, \psi_b \,\rangle.$$

How many applications of Q are required before you can perform a measurement and obtain a correct answer with probability at least $1 - p$?

(f) [4 marks] Provide an informal description of the complexity class NP. Does it matter here that $\mathcal{A}$ solves a problem which is in NP?

2. (a) [3 marks] Two qubits are prepared in one of the four Bell states

$$|\,\Phi_\pm\,\rangle = \tfrac{1}{\sqrt{2}}(|\,00\,\rangle \pm |\,11\,\rangle), \qquad |\,\Psi_\pm\,\rangle = \tfrac{1}{\sqrt{2}}(|\,01\,\rangle \pm |\,10\,\rangle).$$

Show that the Bell states form an orthonormal basis in the Hilbert space associated with two qubits. What does it mean that the Bell states are stabilised by $\pm Z \otimes Z$ and $\pm X \otimes X$? Specify stabiliser generators for each of the four Bell states.

(b) [2 marks] Let S_1 and S_2 be stabiliser generators for a two qubit state $|\,\psi\,\rangle$. The state is modified by a unitary operation U. What are the stabiliser generators for $U|\,\psi\,\rangle$?

(c) [3 marks] Recall that the n-qubit Pauli group is defined as

$$\mathcal{P}_n = \{\mathbb{1}, X, Y, Z\}^{\otimes n} \otimes \{\pm 1, \pm i\}$$

where X, Y, Z are the Pauli matrices. Each element of $\mathcal{P}_n$ is, up to an overall phase ± 1, $\pm i$, a tensor product of Pauli matrices and identity matrices acting on the n qubits. Elements of the Pauli group either commute or anticommute. Show, using stabiliser generators or otherwise, that, up to an overall phase, the elements of $\mathcal{P}_2$ map the Bell states into the Bell states.

(d) [4 marks] Charlie prepares three qubits in the state

$$\frac{1}{\sqrt{2}}\left(|\,000\,\rangle + |\,111\,\rangle\right). \tag{1}$$

He gives one qubit to Alice and one to Bob, and keeps the third one for himself. Trace over the third qubit and show that Alice and Bob share a bipartite state described by the density operator

$$\varrho = \frac{1}{2}|\,\Phi_+\,\rangle\langle\,\Phi_+\,| + \frac{1}{2}|\,\Phi_-\,\rangle\langle\,\Phi_-\,|.$$

Is this an entangled state?

(e) [4 marks] Given the bipartite state ϱ, Alice applies one of the four unitary operations $\{\mathbb{1}, X, Y, Z\}$ to her qubit and sends it to Bob. Can Bob, who performs the measurement in the Bell basis, tell which operation was chosen by Alice? How many bits of information can Alice communicate to Bob?

(f) [5 marks] Charlie, after preparing the state (1) and giving the two qubits to Alice and Bob, applies the Hadamard gate to his qubit and then measures it in the standard basis. He communicates the outcome of the measurement to Bob. Can Bob now tell which of the four operations was chosen by Alice, and if so, how? Does it matter whether Charlie performs his measurement before or after Bob's measurement?

(g) [4 marks] Suppose a third party, who may or may not know the outcome of Bob's measurement, intercepts the qubit that Alice sent to Bob. Explain why there is no measurement that the third party can perform to determine which message Alice transmits.

3. Any density matrix of a single qubit can be parametrised by the three real components of the Bloch vector $\vec{s} = (s_x, s_y, s_z)$ and written as

$$\varrho = \tfrac{1}{2}\left(\mathbb{1} + \vec{s} \cdot \vec{\sigma}\right),$$

where σ_x, σ_y and σ_z are the Pauli matrices, and $\vec{s} \cdot \vec{\sigma} = s_x \sigma_x + s_y \sigma_y + s_z \sigma_z$.

(a) [3 marks] Check that such parametrised ϱ is a density matrix. Explain why the length of the Bloch vector cannot exceed 1.

(b) [5 marks] Any physically admissible operation on a qubit is described by a completely positive map which can always be written as

$$\varrho \mapsto \varrho' = \sum_k A_k \varrho A_k^\dagger \tag{2}$$

where matrices A_k must satisfy

$$\sum_k A_k^\dagger A_k = \mathbb{1}. \tag{3}$$

Show that this map preserves positivity and trace. Show that any composition of completely positive maps is also completely positive.

(c) [9 marks] Any linear transformation T acting on density matrices of a qubit can be completely characterised by its action on the four basis matrices $|\, a \,\rangle\langle\, b \,|$, where $a, b = 0, 1$,

$$T(|\, a \,\rangle\langle\, b \,|) = \sum_{\alpha,\beta=0,1} T_{(\alpha a)(\beta b)} |\, \alpha \,\rangle\langle\, \beta \,|.$$

Using conditions (2) and (3), or otherwise, show that for completely positive maps the 4×4 matrix $T_{(\alpha a)(\beta b)}$ must be positive semidefinite and must satisfy

$$\sum_\alpha T_{(\alpha a)(\alpha b)} = \delta_{ab}, \qquad T^\star_{(\alpha a)(\beta b)} = T_{(\beta b)(\alpha a)}.$$

(d) [8 marks] Let T be defined as,

$$T(\mathbb{1}) = \mathbb{1}, \quad T(\sigma_x) = x\sigma_x, \quad T(\sigma_y) = y\sigma_y, \quad T(\sigma_z) = z\sigma_z.$$

where x, y, z are some real numbers. What is the range of x, y, z for which the map T is positive? Using the matrix representation of T, or otherwise, determine the range for which it is completely positive.

[*The Pauli matrices* $\sigma_x \equiv X$, $\sigma_y \equiv Y$, *and* $\sigma_z \equiv Z$ *are*

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

They anticommute and square to the identity $X^2 = Y^2 = Z^2 = \mathbb{1}$.]