A14687W1

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C7.4
Honour School of Mathematics Part B: Paper C7.4
Honour School of Mathematics and Computer Science Part B: Paper C7.4
Honour School of Mathematics and Statistics Part B: Paper C7.4

INTRODUCTION TO QUANTUM INFORMATION

Trinity Term 2015

THURSDAY, 11 JUNE 2015, 2.30pm to 4.00pm

You may submit answers to as many questions as you wish but only the best two will count for the total mark.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

Do not turn this page until you are told that you may do so

1. The Hadamard transform is defined as

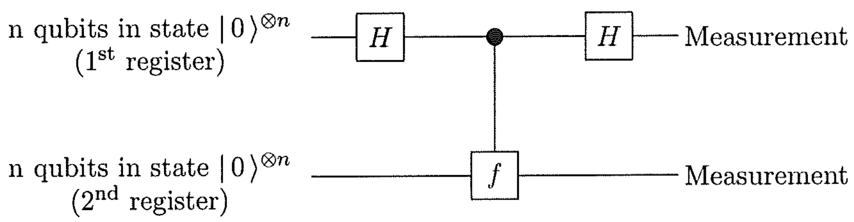
$$| x \rangle \mapsto \frac{1}{\sqrt{2^n}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} | y \rangle, \tag{1}$$

where $x, y \in \{0,1\}^n$ and the operation $x \cdot y$ is defined as

$$x \cdot y = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n \, (\text{mod} 2) \tag{2}$$

(a) [3 marks] Sketch the quantum network which effects the Hadamard transform and explain why it is often useful as the first operation in quantum algorithms.

Let $f : \{0,1\}^n \mapsto \{0,1\}^n$ be a 2-to-1 function such that $f(x+s) = f(x)$, where s is a binary string of length n which is different from zero ($s \neq 0^n$) and $x + s$ is a bit-wise addition modulo 2. In the network below the H operations denote the Hadamard transform on n qubits and the f operation represents a quantum evaluation of f; $| x \rangle | y \rangle \mapsto | x \rangle | y + f(x) \rangle$.



(b) [4 marks] What is the state of the two registers right after the quantum function evaluation?

(c) [5 marks] The second register is measured bit by bit in the computational basis and a binary string $k \in \{0,1\}^n$ is registered. What is the state of the first register after the measurement?

(d) [6 marks] Subsequently the Hadamard transform is performed on the first register, followed by a measurement in the computational basis. The result is a binary string, z. Show that $z \cdot s = 0$.

(e) [7 marks] Suppose the function f is presented as an oracle. How many calls to the oracle are required in order to find s? How does it compare with a classical algorithm for the same problem? Provide rough estimates, detailed derivations are not required.

2. (a) [2 marks] Explain why a self-adjoint (or Hermitian) matrix must be both positive semi-definite and have trace 1 in order to be considered a density matrix.

(b) [7 marks] Consider two qubits in the quantum state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|1\rangle \otimes \left(\sqrt{\frac{2}{3}}|0\rangle + \sqrt{\frac{1}{3}}|1\rangle\right) + |0\rangle \otimes \left(\sqrt{\frac{2}{3}}|0\rangle - \sqrt{\frac{1}{3}}|1\rangle\right)\right]. \quad (3)$$

Use the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ and write explicitly the density matrix ρ of the two qubits corresponding to state $|\psi\rangle$.

(c) [4 marks] Consider operator A on the Hilbert space $\mathcal{H}_A$ and operator B on the Hilbert space $\mathcal{H}_B$. The partial trace over $\mathcal{H}_B$ is defined for the tensor product operators,

$$\mathrm{Tr}_B\, A \otimes B = A\,(\mathrm{Tr}\, B)$$

and extended to any other operator on $\mathcal{H}_A \otimes \mathcal{H}_B$ by linearity. Consider the density operator ρ in (b) and find the reduced density matrices ρ_1 and ρ_2 of the first and the second qubit, respectively.

(d) [5 marks] The trace norm of a matrix A is defined as

$$||A||_{tr} = \mathrm{Tr}\left(\sqrt{A^\dagger A}\right).$$

Show that the trace norm of any self-adjoint matrix is the sum of the absolute values of its eigenvalues. What is the trace norm of a density matrix?

(e) [7 marks] The trace distance between density matrices ρ_1 and ρ_2 is defined as

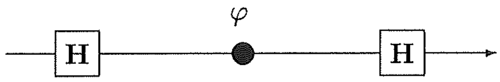
$$T(\rho_1, \rho_2) = \frac{1}{2}||\rho_1 - \rho_2||_{tr}.$$

If a physical system is equally likely to be prepared either in state ρ_1 or state ρ_2 then a single measurement can distinguish between the two preparations with the probability at most

$$\frac{1}{2}\left[1 + T(\rho_1, \rho_2)\right].$$

You are given one of the two, randomly selected, qubits of state $|\psi\rangle$ in Eq. (3). What is the maximal probability with which you can determine whether it is the first or the second qubit?

3. A quantum network which describes a single qubit interference can be represented as follows:



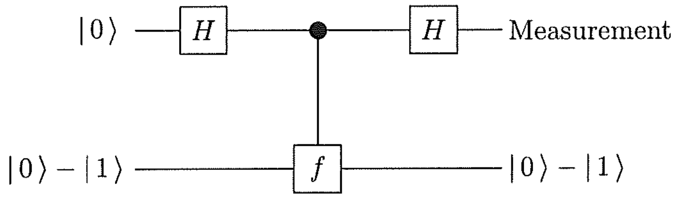
(a) [6 marks] Give the expression for the single-qubit Hadamard gate H in the $\{|0\rangle, |1\rangle\}$ basis. Assume that the qubit is initially in state $|0\rangle$ and find the output state $|\psi_{\text{out}}\rangle$. What is the probability $P_0(\varphi)$ for the qubit to be found in state $|0\rangle$ at the output?

(b) [12 marks] Now, suppose that after the phase gate and before the second Hadamard gate the qubit undergoes decoherence by interacting with an environment in state $|e\rangle$ so that

$$|0\rangle|e\rangle \mapsto |0\rangle|e_0\rangle, \tag{4}$$
$$|1\rangle|e\rangle \mapsto |1\rangle|e_1\rangle, \tag{5}$$

where $|e_0\rangle$ and $|e_1\rangle$ are the new states of the environment which are normalised but not necessarily orthogonal. The decoherence modifies $P_0(\varphi)$ which becomes a function of φ and of the scalar product $\langle e_0|e_1\rangle$. Writing $\langle e_0|e_1\rangle = ve^{i\alpha}$ express P_0 as a function of φ, v, and α. Suppose the decoherence takes place between the first Hadamard gate and the phase gate, how different is the expression for $P_0(\varphi, v, \alpha)$?

(c) [7 marks] Deutsch's algorithm with an oracle $f : \{0,1\} \mapsto \{0,1\}$, is implemented by the following network



Assume that only the first (top) qubit is affected by decoherence as described by Eqs. (4) and (5). How reliably can you tell whether f is constant or balanced?